Math 217 Fall 2025
Quiz 33 – Solutions

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1. Complete[*] the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:
   (a) The linear transformation $T : V \to V$ of a finite dimensional vector space V is *diagonalizable* if ...

   Solution: The linear transformation $T : V \to V$ is *diagonalizable* if there exists a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$, i.e., $[T]_{\mathcal{B}}$ is diagonal.

   (b) Suppose $n \in \mathbb{N}$. An $n \times n$ matrix A is *diagonalizable* if ...

   Solution: An $n \times n$ matrix A is *diagonalizable* if the map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_A(\vec{x}) = A\vec{x}$ is a diagonalizable linear transformation.

   (c) Suppose $m \in \mathbb{N}$. An $m \times m$ matrix A is *orthogonal* if ...

   Solution: An $m \times m$ matrix A is *orthogonal* if its transpose is its inverse; that is,

   $$A^\top A = I_m \quad \text{(equivalently, } AA^\top = I_m\text{)}.$$

   In other words, $A^{-1} = A^\top$, so the columns (and rows) of A form an orthonormal set in $\mathbb{R}^m$.

2. (a) Suppose W and V are vector spaces and $S\colon W \to V$ is a linear transformation. Do eigenvectors and eigenvalues make sense in this context?

   Solution: In general, *no*.
   To talk about eigenvalues and eigenvectors of a linear transformation T, we need to be able to write

   $$T(v) = \lambda v,$$

   where both $T(v)$ and v live in the *same* vector space.
   Here $S : W \to V$ has domain W and codomain V. Unless $W = V$ (and S is a linear transformation from a vector space to itself), the expression

   $$S(w) = \lambda w$$

   does not make sense, because $S(w) \in V$ while $w \in W$ are, in general, elements of different spaces.

[*]For full credit, please write out fully what you mean instead of using shorthand phrases.

So eigenvalues and eigenvectors are defined for linear transformations $T : V \to V$. For a general map $S : W \to V$ with $W \neq V$, the notion of eigenvalue/eigenvector does not apply.

(b) Suppose V is a vector space and let $V \xrightarrow{T} V$ be a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$. If $v_i \in E_{\lambda_i}$ is nonzero, then (v_1, v_2) is linearly independent.

Solution: Recall that

$$E_{\lambda_i} = \{\, v \in V : T(v) = \lambda_i v \,\}, \quad i = 1, 2.$$

Suppose $v_1 \in E_{\lambda_1}$ and $v_2 \in E_{\lambda_2}$ are nonzero and $\lambda_1 \neq \lambda_2$.
Assume to the contrary that (v_1, v_2) is linearly dependent. Since v_1 and v_2 are nonzero and we only have two vectors, linear dependence implies that one vector is a scalar multiple of the other. So there exists $\alpha \neq 0 \in \mathbb{R}$ such that

$$v_2 = \alpha v_1.$$

Apply T to both sides:

$$T(v_2) = T(\alpha v_1) = \alpha T(v_1) = \alpha \lambda_1 v_1.$$

But we also know $v_2 \in E_{\lambda_2}$, so

$$T(v_2) = \lambda_2 v_2 = \lambda_2(\alpha v_1) = \alpha \lambda_2 v_1.$$

Thus

$$\alpha \lambda_1 v_1 = \alpha \lambda_2 v_1.$$

Rewriting,

$$\alpha(\lambda_1 - \lambda_2) v_1 = 0.$$

Since $\lambda_1 \neq \lambda_2$, it implies that $v_1 = 0$, which is a contradition.
Therefore our assumption that (v_1, v_2) is linearly dependent is false, and we conclude that (v_1, v_2) is linearly *independent*.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose V is a vector space. Every linear transformation from V to V is diagonalizable.

Solution: FALSE.
A linear transformation need not have enough eigenvectors to form a basis, so it may fail to be diagonalizable.
Counterexample: Let $V = \mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The only eigenvalue of A is $\lambda = 1$ (the characteristic polynomial is $(1 - \lambda)^2$). Solving

$(A - I)\vec{v} = 0$ gives

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad y = 0,$$

so all eigenvectors have the form $(x, 0)^\top$ with $x \neq 0$. Thus the eigenspace for $\lambda = 1$ is one-dimensional, so there are not enough eigenvectors to form a basis of $\mathbb{R}^2$.

Therefore T is not diagonalizable, showing the statement is false.

(b) Suppose V is a finite-dimensional vector space and $T\colon V \to V$ is a linear transformation. If T is diagonalizable and $\mathcal{A}$ is a basis of V, then $[T]_\mathcal{A}$ is similar to a diagnonal matrix.

Solution: TRUE.

Since T is diagonalizable, there exists a basis $\mathcal{B}$ of V consisting of eigenvectors of T. By definition of diagonalizable, the matrix of T with respect to $\mathcal{B}$ is a diagonal matrix; call it D:

$$[T]_\mathcal{B} = D,$$

where D is diagonal.

Let $S_{\mathcal{B}\to\mathcal{A}}$ be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{A}$ (an invertible matrix). Then the standard change-of-basis formula for matrices of a linear transformation gives

$$[T]_\mathcal{A} = S_{\mathcal{B}\to\mathcal{A}}\,[T]_\mathcal{B}\,S_{\mathcal{B}\to\mathcal{A}}^{-1} = S_{\mathcal{B}\to\mathcal{A}}\,D\,S_{\mathcal{B}\to\mathcal{A}}^{-1}.$$

Thus $[T]_\mathcal{A}$ is similar to the diagonal matrix D.

Hence the statement is true.